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Thermal instability of viscoelastic fluids in horizontal porous layers as initial value problem

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Abstract

The instability of a viscoelastic fluid saturating a horizontal porous layer heated from below is studied theoretically with a dynamical system approach. The viscoelastic character of the flow is taken into account by a modified Darcy's law. The conservation equations of mass, momentum and energy are approximated by a reduced-order system of nonlinear, ordinary differential equations, which is similar to the well-known system derived by Lorenz to describe atmospheric convection. Equilibrium points and their stability are expressed as functions of a dimensionless heat capacity and of two relaxation parameters. Qualitative expressions of the Nusselt number when the system is out of equilibrium are also derived.

Keywords: Rayleigh–Bénard convection; Porous media; Viscoelastic fluids; Dynamical systems; Lorenz attractor

1. Introduction

The study of fluid currents driven by buoyancy forces induced by a temperature gradient has been a classical problem of fluid mechanics for more than one century [1–5]. One interesting variation of this problem was studied by Horton and Rogers [6] and independently by Lapwood [7], who addressed the Rayleigh–Bénard convection in porous media. Beyond its undoubted scientific appeal, buoyancy-driven convection in porous media is relevant to several engineering applications, such as solar energy storage systems, geothermal reservoirs, passive cooling of nuclear reactors, pollutant transport in underground waters, soil decontamination, storage of chemical or agricultural products, as well as many others. Extensive reviews on this subject can be found in the books by Nield and Bejan [8] and by Kaviany [9].

One interesting case of convection in porous media arises when the fluid is viscoelastic. As a matter of fact, viscoelastic flows in porous media have been considered already several years ago [10,11], whereas the study of heat transfer in viscoelastic fluids is relatively more recent [12,13], also because it often leads to problems that can be solved only with the aid of numerical simulations.

Recently, Kim et al. [14] conducted a theoretical analysis of thermal instability driven by buoyancy forces in an initially quiescent, horizontal porous layer saturated by viscoelastic fluids, neglecting the difference between the heat capacity of the fluid and that of the porous matrix. They used the classic linear stability theory to find the critical condition for the onset of convective motion, and the amplitude expansion method to determine the magnitude of fluctuations (and hence the Nusselt number) after departure from equilibrium. A similar linear stability approach was used to investigate the Rayleigh–Bénard problem with non-uniform temperature gradient in a high porosity medium [15].

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Nomenclature

A	parameter in Eq. (49)
a	wavenumber
B	parameter in Eq. (49)
b	parameter of Lorenz's model
C	dimensionless heat capacity, $C = \frac{\rho c}{\varphi\rho c+(1-\varphi)\rho_M c_M}$
c	heat capacity of the fluid
D	dimensionless characteristic time, $D = \frac{\tau\alpha}{h^2}\pi^2(1+a^2)$
$\mathbf{g}$	gravity acceleration
h	porous layer thickness
J	function of characteristic times and heat capacity
k	thermal conductivity of the fluid
Nu	Nusselt number, $Nu = -\frac{h}{\delta T}\frac{\partial T}{\partial y}\Big\vert_{y=0}$
P	pressure
R	Rayleigh–Darcy number
r	normalized Rayleigh–Darcy parameter
T	temperature
t	time
$\mathbf{v}$	velocity
X	dimensionless rate of convective overturning
x	horizontal coordinate
Y	dimensionless temperature variation (horizontal)
y	vertical coordinate
Z	dimensionless temperature variation (vertical)

Greek symbols

α	ratio between the overall thermal conductivity and heat capacity
β	isobaric dilatation coefficient
Γ	permeability
η	dynamic viscosity
φ	porosity
λ	eigenvalue
θ	temperature deviation from the linear, steady-state solution
ρ	fluid density
τ	dimensionless characteristic time
ξ	acceleration coefficient
ω	dimensionless frequency
ψ	stream function
ζ	parameter

A well-known alternative approach to the study of stability is based on the dynamical systems theory, which provides an elegant and straightforward mathematical formalism. Unfortunately, application of this method to most problems of fluid dynamics, which are described by partial differential equations, is not immediate, because their reduction to a dynamical system of ordinary differential equations implies the introduction of infinite degrees of freedom. In some cases, however, it turns out that the system dynamics is dominated only by a few degrees of freedom, which are sufficient to capture the main aspects of the flow evolution. Therefore, it is exactly in these cases that the dynamical system approach becomes extremely useful, for instance, to investigate stability problems.

The best-known example of how the dynamical systems theory can be applied to fluid dynamics is probably represented by Saltzman's approach to the description of atmospheric convection [16], which led Lorentz to formulate his now-famous equations [17] explaining the unpredictability of purely deterministic systems. This approach has been successfully applied to the description of free convection, heat transfer problems, both for Newtonian [18] and, for viscoelastic fluids [19,20].

In the present work, the theory of dynamical systems is used to investigate the thermal instability driven by buoyancy forces in an initially quiescent, horizontal porous layer saturated by viscoelastic fluids. The problem, which is schematically described in Fig. 1, is analogous to the classical problem of thermoconvective instability in simple fluids, and can be described using the same mathematical formalism. The partial differential equations of the perturbed temperature and velocity fields are reduced to a nonlinear system of ordinary differential equations by means of a Galerkin projection, truncated in such a way as to leave only three degrees of freedom [19]. The stability analysis reveals that, depending on the thermophysical properties of the fluid, the onset of the instability may occur for smaller values of the Rayleigh–Darcy number than in the case of a Newtonian fluid.

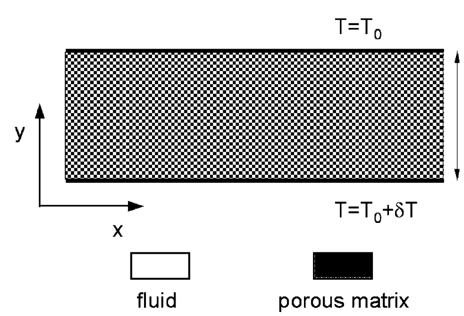

Fig. 1. Schematic description of the problem.

2. Analysis

2.1. Modified Darcy's law

The momentum equation of fluid flow in porous media is usually expressed in the form of Darcy's law [21], which states that for a Newtonian fluid the volume averaged superficial fluid velocity (also known as filtration velocity) is proportional to the pressure gradient driving the flow:

$$\mathbf{v} = -\frac{\Gamma}{\eta}\nabla P \tag{1}$$

where η is the dynamic viscosity of the fluid, and Γ is a quantity called permeability. When gravity must be taken into account, Eq. (1) becomes:

$$\mathbf{v} = -\frac{\Gamma}{\eta}(\nabla P - \rho\mathbf{g}) \tag{2}$$

In general, permeability is a tensor which depends on the porous medium microstructure (shape, size, and orientation of pores), but in the case of isotropy it reduces to a scalar; furthermore, when the Reynolds number is small it is independent of the flow rate and the fluid properties.

Unsteady flows can be described by introducing an inertial term (see the book by Nield and Bejan [8] for a review on the subject):

$$\frac{\xi\rho\Gamma}{\eta}\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v} = -\frac{\Gamma}{\eta}(\nabla P - \rho\mathbf{g}) \tag{3}$$

where ξ is an acceleration coefficient that depends on the geometry.

The use of Darcy's law in viscoelastic flows, however, is not straightforward, because the pressure drop is non-linearly related to the filtration velocity, so that the permeability would also depend on the relaxation time of the fluid. Following a well-established approach [22], one can modify Eq. (3) by including a relaxation term about the pressure gradient, with a characteristic time τ:

$$\frac{\xi\rho\Gamma}{\eta}\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v} = -\frac{\Gamma}{\eta}\left(1 + \tau\frac{\partial}{\partial t}\right)(\nabla P - \rho\mathbf{g}) \tag{4}$$

This equation implicitly assumes that the fluid has a constant viscosity and a single relaxation time which is the case, for instance, of an upper convected Maxwell fluid [23].

In conclusion, the modified Darcy law can be given a synthetic formulation [19] where two characteristic times can be identified:

$$\left(1 + \tau_1\frac{\partial}{\partial t}\right)\mathbf{v} = -\frac{\Gamma}{\eta}\left(1 + \tau_2\frac{\partial}{\partial t}\right)(\nabla P - \rho\mathbf{g}) \tag{5}$$

In particular, τ_1 is the retardation time due to the action of the porous matrix, while τ_2 is the relaxation time depending on viscoelasticity.

2.2. Problem formulation

The problem of free convection in a horizontal layer of a porous medium uniformly heated from below was studied many years ago [6,7]. The departure from equilibrium is governed by the conservation equations of mass, momentum and energy in the Oberbeck–Boussinesq approximation, and assuming local thermal equilibrium between the fluid and the porous medium:

$$\begin{cases} \nabla\cdot\mathbf{v} = 0 \\ \left(1 + \tau_1\frac{\partial}{\partial t}\right)\mathbf{v} = -\frac{\Gamma}{\eta}\left(1 + \tau_2\frac{\partial}{\partial t}\right)(\nabla P - \rho\mathbf{g}) \\ \frac{\partial T}{\partial t} + C(\mathbf{v}\cdot\nabla)T = \alpha\nabla^2 T \end{cases} \tag{6}$$

with

$$\rho = \rho_0[1 - \beta(T - T_0)] \tag{7}$$

$$\tau_1 = \frac{\xi\rho_0\Gamma}{\eta} \tag{8}$$

In Eqs. (6)–(8), T denotes temperature, and β is the isobaric dilatation coefficient. C is the ratio of the volumetric specific heat of the fluid to the overall thermal capacity:

$$C = \frac{\rho c}{\varphi\rho c + (1-\varphi)\rho_M c_M} \tag{9}$$

where c is the specific heat, φ is the matrix porosity (ratio of the volume occupied by the fluid to the total volume), and the subscript M denotes the thermophysical properties of the porous matrix. Finally, α represents the ratio of the overall thermal conductivity to the overall heat capacity:

$$\alpha = \frac{\varphi k + (1-\varphi)k_M}{\varphi\rho c + (1-\varphi)\rho_M c_M} \tag{10}$$

where k indicates the thermal conductivity of the fluid.

In a two-dimensional layer of infinite length and thickness h, the problem is completed by the following boundary conditions for the velocity:

$$\begin{cases} \mathbf{v}(x,0) = 0 \\ \mathbf{v}(x,h) = 0 \end{cases} \quad \forall t \tag{11}$$

and for the temperature field:

$$\begin{cases} T(x,0) = T_0 + \delta T \\ T(x,h) = T_0 \end{cases} \quad \forall t \tag{12}$$

Using a standard notation, [4,5] the temperature distribution inside the layer can be written as

$$T(x,y,t) = T_0 + \delta T\left(1 - \frac{y}{h}\right) + \theta(x,y,t) \tag{13}$$

where $\theta(x,y,t)$ represents the deviation from the linear temperature distribution obtained in case of simple conductive heat transfer in the layer.

In a stream function formulation, conservation of mass given by the first of Eq. (6) is automatically satisfied, and the linearized problem for temperature and velocity perturbations reduces to

$$\left(\tau_1\frac{\partial}{\partial t} + 1\right)\nabla^2\Psi = \frac{\Gamma\beta g\rho_0}{\eta}\left(\tau_2\frac{\partial}{\partial t} + 1\right)\frac{\partial\theta}{\partial x} \tag{14}$$

$$\frac{\partial\theta}{\partial t} + C\left[\frac{\partial\Psi}{\partial y}\frac{\partial\theta}{\partial x} - \frac{\partial\Psi}{\partial x}\frac{\partial\theta}{\partial y} - \frac{\delta T}{h}\frac{\partial\Psi}{\partial x}\right] = \alpha\nabla^2\theta \tag{15}$$

with the following boundary conditions:

$$\left.\frac{\partial \Psi}{\partial x}\right|_{y=0} = \left.\frac{\partial \Psi}{\partial x}\right|_{y=h} = 0 \tag{16}$$

$$\left.\frac{\partial \Psi}{\partial y}\right|_{y=0} = \left.\frac{\partial \Psi}{\partial y}\right|_{y=h} = 0 \tag{17}$$

$$\theta(x,0) = \theta(x,h) = 0 \tag{18}$$

The problem can be conveniently put into a dimensionless form rescaling lengths by h, time by h^2/α, and temperature by δT. Consequently, Eqs. (14) and (15) become:

$$\left(\hat{\tau}_1 \frac{\partial}{\partial t} + 1\right)\nabla^2 \Psi = R\left(\hat{\tau}_2 \frac{\partial}{\partial t} + 1\right)\frac{\partial \theta}{\partial x} \tag{19}$$

$$\frac{\partial \theta}{\partial t} + C\left[\frac{\partial \Psi}{\partial y}\frac{\partial \theta}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial \theta}{\partial y} - \frac{\partial \Psi}{\partial x}\right] = \nabla^2 \theta \tag{20}$$

The dimensionless group R is the so-called Rayleigh–Darcy number, defined as

$$R = \frac{\Gamma \beta g \rho_0 h \delta T}{\eta \alpha} \tag{21}$$

while $\hat{\tau}_1$ and $\hat{\tau}_2$ indicate the dimensionless retardation and relaxation time, respectively.

2.3. Model reduction

The standard techniques to investigate the departure from equilibrium of a physical system described by partial differential equations are based on perturbation analysis. An alternative approach consists in studying the stability of the associated dynamical system. The simplest way to reduce a system of partial differential equations to a low order system of ordinary differential equations is to assume that the state of the system can be represented by a finite linear combination of basis functions, or mode shapes, at every instant of time. The coefficient of each basis element in the linear combination is then described by an ordinary differential equation. The key problem of this procedure (which is but a Galerkin projection in the space of eigenfunctions) is choosing a basis that will have the lowest number of elements and yet reproduce the system dynamics accurately. Following Saltzman [16] and Lorenz [17], one can replace the perturbations Ψ and θ with a truncated series of sinusoidal eigenfunctions [19]:

$$\Psi = \frac{\sqrt{2}(1+a^2)}{aC} X(t)\sin(\pi a x)\sin(\pi y) \tag{22}$$

$$\theta = \frac{R^*}{\pi R}\left[\sqrt{2}Y(t)\cos(\pi a x)\sin(\pi y) - Z(t)\sin(2\pi y)\right] \tag{23}$$

where time dependence is included in the variables X, Y and Z, and a is a dimensionless wavenumber. The dimensionless number $R^* = \pi^2(1+a^2)^2/a^2$ is the critical value of the Rayleigh number obtained for Newtonian fluids.

While sinusoids correctly account for the eigenfunction dependence on the homogeneous directions [24], this choice is only an approximation in the non-homogeneous, y-direction.

Substitution of Eqs. (22) and (23) into Eqs. (19) and (20) yields the following system of nonlinear ordinary differential equations:

$$\begin{aligned}
\dot{X} &= -2C\frac{D_2}{D_1}XZ - \frac{1 - rCD_2}{D_1}X + \frac{C(1-D_2)}{D_1}Y \\
\dot{Y} &= -2XZ + rX - Y \\
\dot{Z} &= XY - bZ
\end{aligned} \tag{24}$$

where time has been further rescaled by $\pi^2(1+a^2)$ and

$$D_1 = \hat{\tau}_1 \pi^2(1+a^2) \tag{25}$$

$$D_2 = \hat{\tau}_2 \pi^2(1+a^2) \tag{26}$$

$$b = \frac{4}{1+a^2} \tag{27}$$

$$r = \frac{R}{R^*} \tag{28}$$

The system given by Eqs. (24) is similar to the well-known Lorenz equations [17], the only differences being in the first equation, which has an additional nonlinear term and different coefficients. Like in Lorenz equations, the variable X can be interpreted as the rate of convective overturning, while the variables Y and Z measure the horizontal and the vertical temperature variations, respectively. In spite of their apparent simplicity, these nonlinear dynamical systems are extremely complex, and represent an almost unlimited source of discussion topics for mathematicians [25]. In particular, their most fascinating feature is that when the coefficients of the state variables lay within certain ranges of values, the trajectory of the system in the phase space remains confined inside a finite volume without describing a periodic or quasi-periodic orbit, creating a geometric object which is known as strange attractor. For the Lorenz system, this chaotic behavior can be easily obtained by tuning the value of the Rayleigh number [25].

However, when these systems are generated as simplified mathematical models of physical phenomena, they almost never give a realistic, quantitative description of physical systems far from equilibrium, and therefore strange attractors should be considered at most from a merely qualitative standpoint. Here, the study of Eq. (24) is limited to what is necessary to understand the conditions of stability and the onset of convection in our physical system.

3. Stability

3.1. Critical points

The critical (or equilibrium) points correspond to the steady-state solutions of the dynamical system $\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X})$ defined in Eq. (24), from which they are obtained by setting time derivatives equal to zero. Setting $\mathbf{X} = \{X; Y; Z\}$, these solutions are

$$\mathbf{X}_1 = \{0;\quad 0;\quad 0\} \tag{29}$$

$$\mathbf{X}_2 = \left\{ \sqrt{\tfrac{b}{2}(rC-1)};\quad \tfrac{1}{C}\sqrt{\tfrac{b}{2}(rC-1)};\quad \tfrac{1}{2C}(rC-1) \right\} \tag{30}$$

$$\mathbf{X}_3 = \left\{ -\sqrt{\tfrac{b}{2}(rC-1)};\quad -\tfrac{1}{C}\sqrt{\tfrac{b}{2}(rC-1)};\quad \tfrac{1}{2C}(rC-1) \right\} \tag{31}$$

The two symmetric equilibrium points given by Eqs. (30) and (31) are real only if $r < 1/C$, that is

$$R \geqslant \frac{\pi^2(1+a^2)^2}{Ca^2} \tag{32}$$

This threshold is independent of all relaxation parameters and, for $C = 1$, corresponds to the well-known bifurcation of the solution obtained for Newtonian fluids.

It must be remarked that these are critical points of the reduced-order mathematical model described by Eq. (24), and therefore they might not represent adequately their counterparts in the physical system. In particular, the equilibrium point $\mathbf{X}_1$ does correspond both qualitatively and quantitatively to its counterpart (that is, the situation for which the fluid is motionless and heat transfer through the porous layer is purely conductive), whereas $\mathbf{X}_2$ and $\mathbf{X}_3$ represent two symmetric situations of steady-state convection only from a qualitative standpoint, and nothing can be said about their quantitative agreement with the physical system.

3.2. Linearized system around equilibria

In order to investigate the stability of critical points, one can linearize the dynamical system $\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X})$ in the neighborhood of each of them, by taking the Jacobian of the vectorial function $\mathbf{f}(\mathbf{X})$, and evaluating it for $\mathbf{X} = \mathbf{X}_1$, $\mathbf{X} = \mathbf{X}_2$, $\mathbf{X} = \mathbf{X}_3$, respectively:

$$\frac{\partial \mathbf{f}}{\partial \mathbf{X}} = \begin{pmatrix} -2C\frac{D_2}{D_1}Z - \frac{1-rCD_2}{D_1} & \frac{C(1-D_2)}{D_1} & -2C\frac{D_2}{D_1}X \\ -2Z + r & -1 & -2X \\ Y & X & -b \end{pmatrix} \tag{33}$$

Without entering the details of stability theory, one can simply state that the system is asymptotically stable in the neighborhood of a given equilibrium point if and only if all the eigenvalues of the corresponding Jacobian have a negative real part, whereas if there is at least one eigenvalue with a positive real part the system is unstable.

The characteristic equation obtained for $\mathbf{X} = \mathbf{X}_1$ is

$$(\lambda + b)\left(\lambda^2 + \frac{D_1 + 1 - rCD_2}{D_1}\lambda + \frac{1 - rC}{D_1}\right) = 0 \tag{34}$$

the character of the roots of Eq. (34) can be determined thanks to Hurwitz's stability criterion, which states that Eq. (34) has three roots with negative real part if and only if:

$$r < \min\left\{J_0; \frac{1}{C}\right\} \tag{35}$$

where J_0 is a function of both characteristic times and of the dimensionless heat capacity:

$$J_0 = \frac{1 + D_1}{CD_2} = \frac{1 + \pi^2(1+a^2)\hat{\tau}_1}{C\pi^2(1+a^2)\hat{\tau}_2} \tag{36}$$

The smallest critical value of the Rayleigh–Darcy number for the onset of convection is obtained by minimizing the stability condition with respect to a. In particular, if $J_0 > 1$ one finds $a_c = 1$, which gives $R_c = 4\pi^2/C$, whereas if $J_0 < 1$ minimization of Eq. (36) yields $a_c^2 = \sqrt{1 + 1/\pi^2\hat{\tau}_1}$ and

$$R_c = \frac{1 + \sqrt{1 + 1/\pi^2\hat{\tau}_1}\left[1 + \pi^2\hat{\tau}_1(\sqrt{1 + 1/\pi^2\hat{\tau}_1})\right]}{C\hat{\tau}_2\sqrt{1 + 1/\pi^2\hat{\tau}_1}} \tag{37}$$

The solutions $\mathbf{X} = \mathbf{X}_2$ and $\mathbf{X} = \mathbf{X}_3$ lead to the same characteristic equation:

$$\lambda^3 + \left(\frac{1}{D_1} + b + 1 - \frac{D_2}{D_1}\right)\lambda^2 + b\left[\frac{1}{D_1} + rC + \frac{D_2}{D_1}(rC - 2)\right]\lambda + \frac{2b(rC-1)}{D_1} = 0 \tag{38}$$

Applying Hurwitz's criterion again, one finds that these equilibrium points are asymptotically stable if and only if the following inequalities are satisfied simultaneously:

$$J_1 = 1 + (1+b)D_1 - D_2 > 0 \tag{39}$$

$$J_2 = rC[J_1(D_1 + D_2) - 2D_1] + J_1(1 - 2D_2) + 2D_1 > 0 \tag{40}$$

The analysis of Eqs. (35), (39) and (40) allows one to determine the number and the conditions for the existence of steady-state, asymptotically stable solutions to Eqs. (24). Table 1 summarizes all possible combinations of the parameters J_0, J_1 and J_2, and the corresponding stable solutions of the system: the condition $J_0 > 1$ is not compatible with the condition $J_1 < 0$, so that this combination is not allowed.

Table 1
Parameter combinations resulting into steady-state, stable solutions of Eq. (17)

Parameter			Number of asymptotically stable solutions
J_0	J_1	J_2	
$<1/C$	>0	>0	1 for $r < J_0$ 0 for $J_0 < r < 1/C$ 2 for $r > 1/C$
$<1/C$	>0	<0	
$<1/C$	<0	>0	1 for $r < J_0$
$<1/C$	<0	<0	0 for $r > J_0$
$>1/C$	>0	>0	1 for $r < 1/C$ 2 for $r > 1/C$
$>1/C$	>0	<0	1 for $r < 1/C$ 0 for $r > 1/C$
$>1/C$	<0	>0	Impossible
$>1/C$	<0	<0	

3.3. Parametric effects

Although the system stability around critical points is fully characterized by Eqs. (34) and (37), some further analysis is required to unfold the role played by the different physical parameters involved in these equations, which are the dimensionless heat capacity, C, and the retardation and relaxation parameters, D_1 and D_2. In particular, stability of the critical point $\mathbf{X} = \mathbf{X}_1$ is of special interest, because it determines the onset of convection.

Eqs. (35), (36) and (40) show that the dimensionless heat capacity has a destabilizing effect on the system, that is, transitions occur at smaller Rayleigh–Darcy numbers when C is large. This can be understood qualitatively by recalling how C has been defined in Eq. (9), and the hypothesis of thermal equilibrium between the fluid and the porous matrix, which is justified because flows through porous media are usually very slow. Small values of C mean that the heat capacity of the porous matrix is very large: thus, the matrix cools down the fluid by absorbing its thermal energy, and leaves no energy available to allow escaping from equilibrium. On the contrary, if C is large all thermal energy will stay confined in the fluid, facilitating the onset of unstable motions. To avoid unnecessary complications, in the rest of the paragraph the effects of other parameters will be studied in the case $C = 1$ without loss of generality.

The role of retardation and relaxation parameters is somewhat more complicated, and needs a more detailed explanation. For Newtonian fluids the instability is known to occur for $r = 1$, whereas for viscoelastic fluids, according to Eq. (35), this is no longer true when $D_2 - D_1 > 1$, and the critical value becomes smaller. In terms of physical quantities this means that, in a certain range of parameters D_1 and D_2, that is, of the retardation time $\hat{\tau}_1$ and the relaxation time $\hat{\tau}_2$, the critical Rayleigh–Darcy number for a viscoelastic fluid is smaller than that for a Newtonian fluid. Fig. 2 shows the neutral stability curves as a function of the wavenumber a, for different values of the characteristic times satisfying the above condition, and compares them with the curve obtained for a Newtonian fluid, which is obtained by setting $r = 1$. The minimum value of the Rayleigh–Darcy number on each curve marks the onset of instability, and for the Newtonian case one finds again $a = 1$ and $R = 4\pi^2$. Critical values of r for the onset of convection in a quiescent viscoelastic fluid (Eq. 35) are plotted in Fig. 3, with respect to parameters D_2 and D_1.

The qualitative behavior of the system in the neighborhood of critical points can be obtained by studying the roots of the corresponding characteristic equations. In particular, for the equilibrium $\mathbf{X} = \mathbf{X}_1$ one finds three solutions:

$$\lambda_{1,2} = \frac{1}{2D_1}\left(rD_2 - D_1 - 1 \pm \sqrt{r^2D_2^2 - 2rD_1D_2 - 2rD_2 + D_1^2 - 2D_1 + 1 + 4rD_1}\right)$$
$$\lambda_3 = -b \tag{41}$$

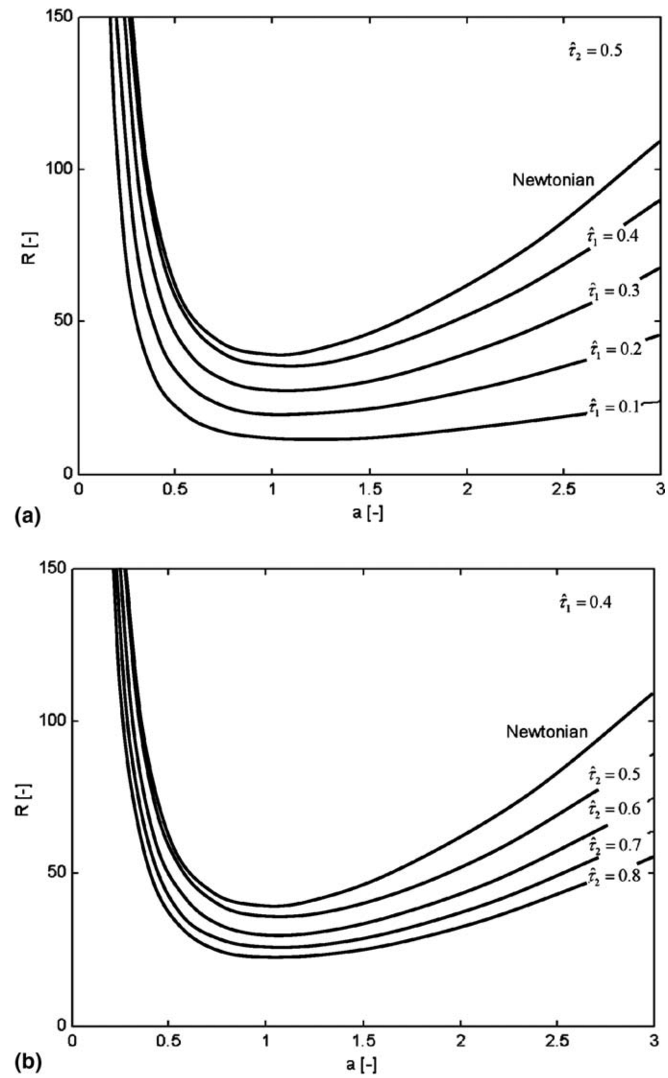

Fig. 2. Neutral stability curves showing the critical Rayleigh–Darcy number as a function of wavenumber, a, and their dependence on (a) retardation time and (b) relaxation time.

Since λ_3 is always negative, the corresponding eigenvector represents a stable manifold of the phase space, linearized around $\mathbf{X} = \mathbf{X}_1$: points of the phase space belonging to this manifold converge to the equilibrium point, the rate of convergence being given by $\exp(\lambda_3)$. On the contrary, solutions λ_1 and λ_2 lead to different behaviors according to the values of parameters.

The loci of roots λ_1 and λ_2 are plotted on the complex plane as a function of r in Figs. 4 and 5, for $D_2 - D_1 < 1$ and $D_2 - D_1 > 1$, respectively. In the former case, eigenvalues can be either negative or positive, but are always on the real axis: thus, their eigenvectors are tangent to a stable manifold ($\lambda < 0$) or to an unstable one ($\lambda > 0$). In particular, when both eigenvalues are negative the critical point $\mathbf{X} = \mathbf{X}_1$ is a stable sink, and when the largest one becomes positive it changes to a saddle point, with two stable and one unstable manifolds.

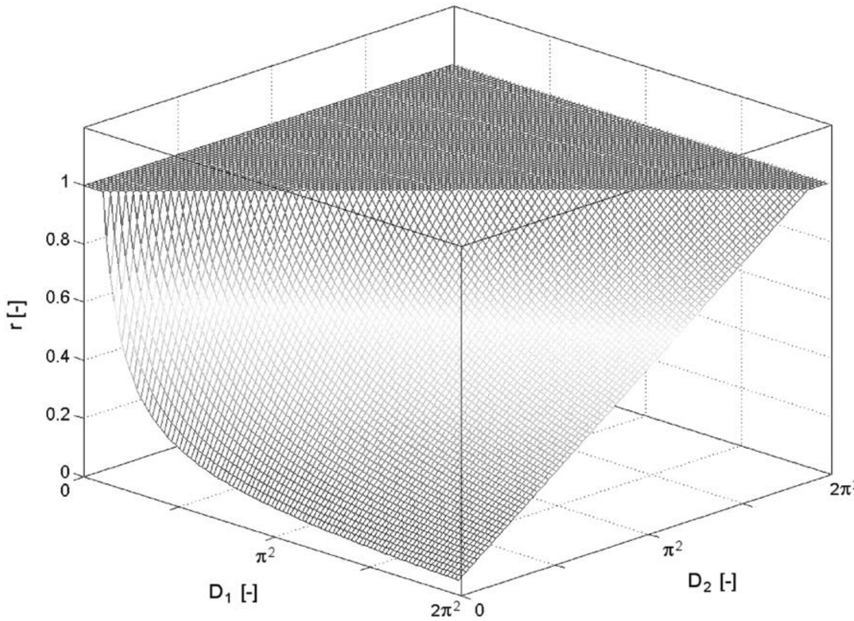

Fig. 3. Critical Rayleigh–Darcy number in the domain of the retardation parameter, D_1, and the relaxation parameter, D_2.

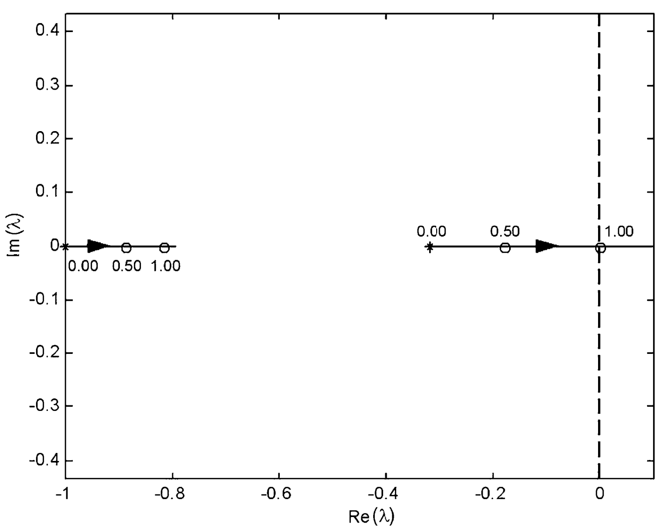

Fig. 4. Position on the complex plane of the eigenvalues λ_2 and λ_3 of the linearized system around the equilibrium point $\mathbf{X}_1$ (Eq. 34), as a function of parameter r (root locus). $D_1 = \pi^2$, $D_2 = \pi^2/2$, $0 < r < 1.25$.

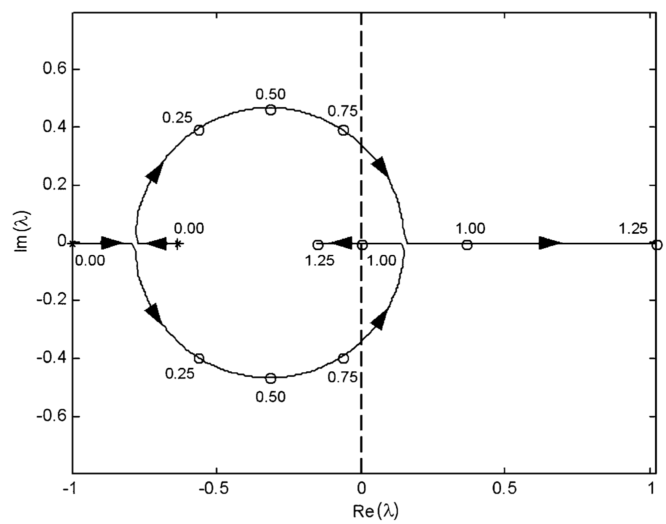

Fig. 5. Position on the complex plane of the eigenvalues λ_2 and λ_3 of the linearized system around the equilibrium point $\mathbf{X}_1$ (Eq. 34), as a function of parameter r (root locus). $D_1 = \pi^2/2$, $D_2 = \pi^2$, $0 < r < 1.25$.

If $D_2 - D_1 > 1$, Eq. (34) yields a pair of complex conjugate eigenvalues for:

$$\frac{1 + D_1 - 2\frac{D_1}{D_2} - 2\sqrt{D_1(D_2 - D_1)(D_2 - 1)}}{D_2} < r < \frac{1 + D_1 - 2\frac{D_1}{D_2} + 2\sqrt{D_1(D_2 - D_1)(D_2 - 1)}}{D_2} \quad (42)$$

as shown in Fig. 5. In this case, the critical point is a classified as focus, and trajectories in phase space follow a spiraling path, converging towards the focus if it is stable (that is, if the eigenvalues have negative real part) or diverging if it is unstable. The qualitative topology of the manifold generated by λ_1 and λ_2, is described schematically in Fig. 6 for values of r representative of the different cases.

4. Departure from equilibrium

4.1. Steady-state convection

The behavior of the system when the Rayleigh–Darcy number grows beyond the critical value, and convective motion of the fluid starts, is of special interest for applications. In particular, one can calculate how much the heat flux through the porous layer increases with respect to

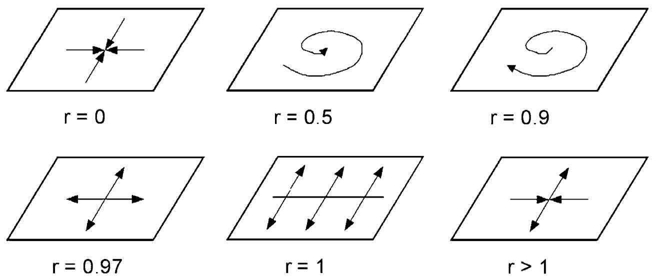

Fig. 6. Qualitative behavior of the system in the neighborhood of the equilibrium point ($X = 0$; $Y = 0$; $Z = 0$), on the manifold of phase space defined by the eigenvectors of λ_2 and λ_3, for different values of parameter r.

the case of pure conduction, which is given by Nusselt's number. To study the departure from equilibrium, one should integrate Eq. (24) with respect to time with an initial condition infinitesimally close to the critical point $\mathbf{X} = \mathbf{X}_1$, and follow the evolution of the system in the phase space. It must be remarked once more that Eq. (24) is a truncated series expansion representing a drastic simplification of the physical system: thus, it adequately describes states where the physical variables (e.g. velocity) are small, but can give considerable errors as variables become larger.

The simplest case of departure from equilibrium occurs when $r > 1/C$. In fact, if this condition is satisfied, the critical points $\mathbf{X}_2$ and $\mathbf{X}_3$ are real: therefore, the system will evolve until it reaches a new steady-state condition defined by either Eq. (30) or Eq. (31). An example of the trajectory described by the system in the phase space is illustrated in Fig. 7.

The dimensionless heat transfer coefficient, or Nusselt number, can be obtained from an energy balance:

$$Nu \frac{\delta T}{h} = -\frac{\partial T}{\partial y}\bigg|_{y=0} \tag{43}$$

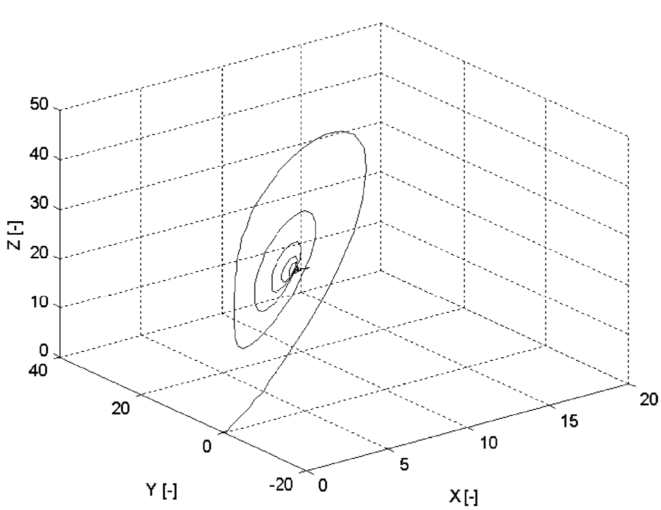

Fig. 7. Trajectory of the system in phase space, when a small perturbation around the equilibrium point ($X = 0$; $Y = 0$; $Z = 0$) is applied, and $r > 1$: the system converges to one of the equilibrium points defined by Eq. (30) or (31). Eq. (24) was solved numerically (Adams' scheme) for $0 < t^* < 1000$, $r = 50$, $D_1 = \pi^2$, and $D_2 = \pi^2/2$, where $t^* = \alpha t/\pi^2 h^2(1 + a^2)$, with initial condition ($X = 0.1$; $Y = 0.1$; $Z = 0.1$).

that is

$$Nu = 1 - \frac{h}{\delta T}\frac{\partial\theta}{\partial y}\bigg|_{y=0} \tag{44}$$

recalling Eq. (23) yields:

$$Nu = 1 - \frac{1}{r}\left[\sqrt{2}Y(t)\cos(\pi a x) - 2Z(t)\right] \tag{45}$$

The oscillating term in Eq. (45) can be deleted by averaging over a finite number of periods, so that substituting for Z the value by Eq. (30) or (31) gives:

$$Nu = 1 + \frac{r-1}{r} \tag{46}$$

where $C = 1$ for simplicity.

Of course, one cannot expect this result to be accurate for large values of r, because it has been drawn from the drastically simplified model given by Eq. (24). Nevertheless, Eq. (46) should be sufficiently precise around the critical value of the Rayleigh–Darcy number, that is, when the system is not too far from the equilibrium point $\mathbf{X} = \mathbf{X}_1$, and temperature and velocity fluctuations are small. Expanding the term $(r-1)/r$ into Taylor's series around $r = 1$ yields:

$$Nu \cong 1 + (r-1)\lfloor 1 - (r-1) + (r-1)^2 - (r-1)^3 + \cdots \rfloor \tag{47}$$

and for $r \approx 1$ one obtains:

$$Nu \sim r \tag{48}$$

This relation suggests a linear dependence of Nusselt's number on the Rayleigh–Darcy number, as it has been found for Newtonian fluids, in the same limit $r \approx 1$ [26]. A similar result can be obtained with the method of amplitude expansion [14] which, however, is tremendously more tedious than the present approach.

4.2. Stable orbit

The picture becomes more complicate when $J_0 < 1/C$, and $r > J_0$. In this case, the critical point $\mathbf{X} = \mathbf{X}_1$ is unstable, but the system cannot converge to another equilibrium point because $\mathbf{X}_2$ and $\mathbf{X}_3$ are not real. As a matter of fact, numerical simulations carried out for this range of parameters indicate the existence of a stable orbit that attract all trajectories leaving point $\mathbf{X} = \mathbf{X}_1$ when the system is given a small perturbation, as shown in the example of Fig. 8. Finding stable orbits of a given dynamical system and their location in the phase space is a non-trivial task even for experienced mathematicians. In this particular case, though, simple geometrical considerations allow one finding the orbit equations.

First of all, one can observe that by definition a stable orbit must be a periodic solution of Eq. (24), with period $\Sigma = 2\pi/\omega$, where ω is the imaginary part of the complex conjugate root pair of Eq. (34). Moreover, the qualitative shapes of the three-dimensional loop projected on the coor-

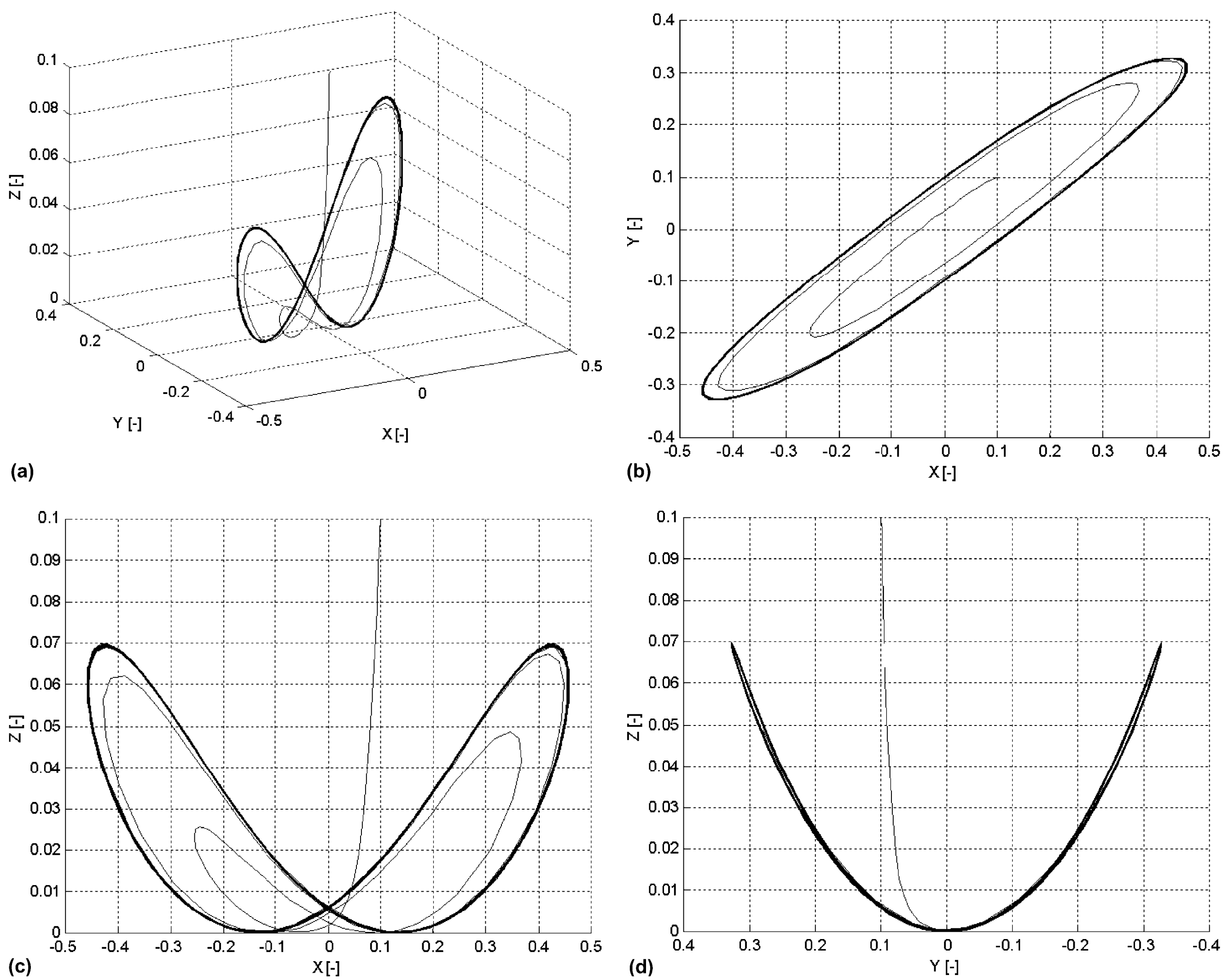

Fig. 8. Trajectory of the system in phase space, when a small perturbation around the equilibrium point ($X = 0$; $Y = 0$; $Z = 0$) is applied, and $J_0 < r < 1$. In this case the system converges to a limit cycle, which is shown (a) in XYZ space, (b) in XY plane, (c) in XZ plane, (d) in YZ plane. Eq. (24) was solved numerically (Adams' scheme) for $0 < t^* < 1000$, $r = 0.8$, $D_1 = \pi^2/2$, and $D_2 = \pi^2$, where $t^* = \alpha t/\pi^2 h^2(1 + a^2)$, with initial condition ($X = 0.1$; $Y = 0.1$; $Z = 0.1$).

dinate planes, which are shown in Fig. 8b–d, belong to the well-known family of Bowditch–Lissajous curves [27], which have the general parametric equations:

$$\begin{aligned} x &= A\sin(nt + \varsigma) \\ y &= B\sin(t) \end{aligned} \tag{49}$$

The combination of these three curves provides unique values for pulsation and phase shift, so that one obtains the following three-dimensional orbit [28]:

$$\begin{aligned} X &= A_1 \sin\left(\omega t + \frac{\pi}{8}\right) \\ Y &= A_2 \sin(\omega t) \\ Z &= A_3 \sin\left(2\omega t + \frac{\pi}{2}\right) \end{aligned} \tag{50}$$

where A_1, A_2 and A_3 are unknown parameters, which can be obtained by introducing Eq. (50) into Eq. (24) and setting again $C = 1$.

The Nusselt number is given again by Eq. (45), so that after space averaging the only relevant variable is Z:

$$Nu = 1 + \frac{2}{r}Z = 1 + \frac{2}{r}A_3 \sin\left(2\omega t + \frac{\pi}{2}\right) \tag{51}$$

where

$$A_3 \sin\left(2\omega t + \frac{\pi}{2}\right) = \frac{1}{2}\left\{ r - \frac{\left[D_1\omega + \tan\left(\omega t + \frac{\pi}{8}\right)\right]\left[\omega + \tan(\omega t)\right]}{\tan\left(\omega t + \frac{\pi}{8}\right)\left[D_2\omega + \tan(\omega t)\right]} \right\} \tag{52}$$

Unlike in the previous case, Z is a function of time, so that the average value over a period must be considered in order to eliminate time dependence. Following the same procedure described above, one can expand the right hand side of Eq. (51) into Taylor's series around the critical value $r = r_0$, where $r_0 = (1 + D_1)/D_2$ marks the onset of convection for this range of parameters. The result, in the limit $r \approx r_0$, is

$$Nu \cong 1 + \frac{r_0 - \Omega}{r_0} + \frac{\Omega}{r_0^2}(r - r_0) \tag{53}$$

where

$$\Omega = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{\left[D_1\omega + \tan\left(\omega t + \frac{\pi}{8}\right)\right]\left[\omega + \tan(\omega t)\right]}{\tan\left(\omega t + \frac{\pi}{8}\right)\left[D_2\omega + \tan(\omega t)\right]}\mathrm{d}t \tag{54}$$

Eq. (53) suggests that in this case the Nusselt number at the onset of convection is still a straight line when plotted as a function of the Rayleigh–Darcy number, but both the slope and the intercept now depend on the retardation and relaxation parameters.

5. Conclusions

The onset of buoyancy-driven motion in a horizontal porous layer saturated with a relaxational fluid was investigated theoretically by means of dynamical system theory. The general problem of buoyancy-driven convection in porous media has a wide variety of engineering applications, such as geothermal reservoirs, agricultural product storage systems, packed-bed catalytic reactors, the pollutant transport in underground and the heat removal of nuclear power plants. Specific examples in which the fluid viscoelasticity should be taken into account include geophysical flows, oil reservoirs, and chemical reactors.

The physical system was described with a reduced-order model, which was generated by expanding temperature and velocity fluctuations into a truncated series of eigenfunctions. The approach is identical to that used by Lorenz to obtain a reduced-order model of atmospheric convection, and shows the unpredictability of deterministic systems.

According to the values of the retardation and relaxation parameters, two different behaviors can be observed. In one case there is a supercritical instability leading to steady-state convection, similar to what has been found for Newtonian fluids, which does not depend on characteristic times. Alternatively, the system trajectories in the phase space collapse onto a limit cycle, so that convective motion exhibits periodic oscillations; the critical Rayleigh–Darcy number for the onset of convection can be much smaller than in the previous case.

Although the simplifications on which the present approach is based are too drastic to allow a quantitative investigation of the physical system far from equilibrium, the reduced-order convection model gives a correct qualitative description of the system behavior. In particular, with simple geometrical considerations and elementary calculus, an estimation of Nusselt's number at the onset of convective motions is possible.

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